

邊界積分方程式應用於 Helmholtz 方程式

函數 $f(x, y)$ 滿足下列 Helmholtz 方程式

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + k^2 f = 0$$

依 Helmholtz 方程式基本解應用所述，對 $f(x, y)$ 的基本解為 $-\frac{1}{4} H_0^{(1)}(kr)$ 。將邊界線以 m 個一定元素加以離散，邊界面 f 與其法線方向導函數 $\bar{f} (= \partial f / \partial n)$ 的關係式如下



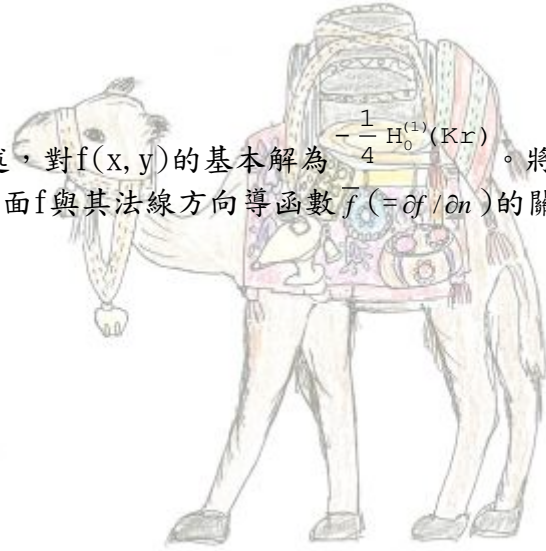
$F = K \bar{F}$

$K = H^{-1} G$

$$H = H_{ij} = \begin{cases} H_{ij} & i \neq j \\ H_{ij} + \frac{1}{2} & i = j \end{cases}$$

$G = G_{ij}$

$i, j = 1, 2, \dots, m$



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H_{ij} 及 G_{ij} 數值計算如下

$i \neq j$ 時

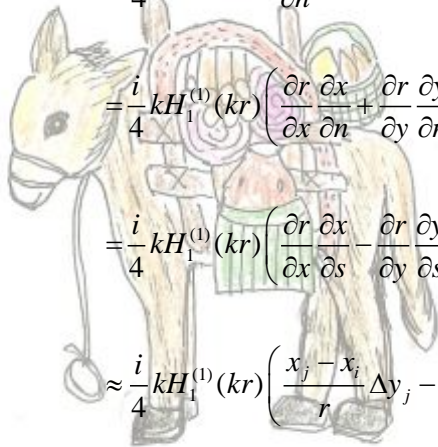
$$H_{ij} = \int_{\Gamma_j} \frac{\partial}{\partial n} \left(-\frac{1}{4} H_0^{(1)}(kr) \right) d\Gamma$$

$$\approx \frac{i}{4} k H_1^{(1)}(kr) \cdot \frac{\partial r}{\partial n} \Gamma_j$$

$$= \frac{i}{4} k H_1^{(1)}(kr) \left(\frac{\partial r}{\partial x} \frac{\partial x}{\partial n} + \frac{\partial r}{\partial y} \frac{\partial y}{\partial n} \right) \Gamma_j$$

$$= \frac{i}{4} k H_1^{(1)}(kr) \left(\frac{\partial r}{\partial x} \frac{\partial x}{\partial s} - \frac{\partial r}{\partial y} \frac{\partial y}{\partial s} \right) \cdot \Gamma_j$$

$$\approx \frac{i}{4} k H_1^{(1)}(kr) \left(\frac{x_j - x_i}{r} \Delta y_j - \frac{y_j - y_i}{r} \Delta x_j \right)$$



$$G_{ij} = \int_{\Gamma_j} \left(-\frac{1}{4} H_0^{(1)}(kr) \right) d\Gamma \approx -\frac{1}{4} H_0^{(1)}(kr) \cdot \Gamma_j$$

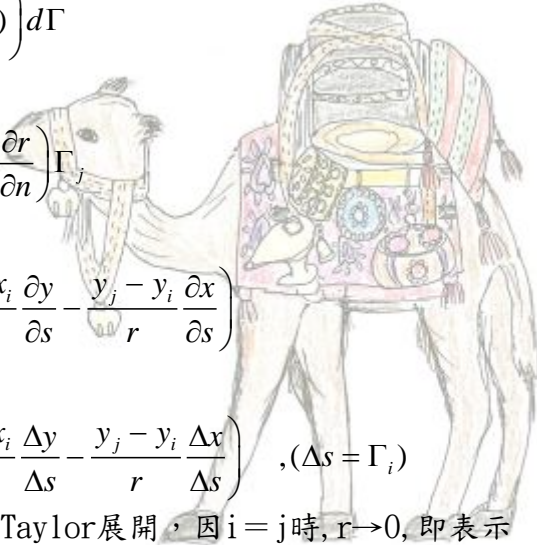
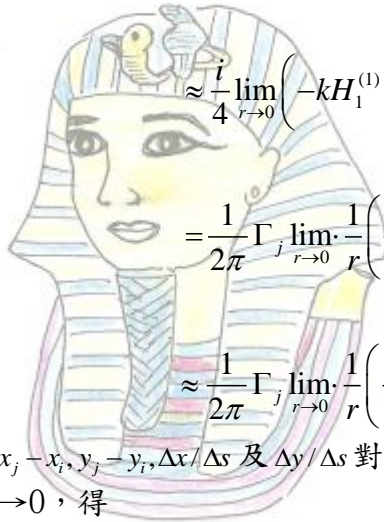
s 表示切線

$$\left. \begin{aligned} \Delta x_j &= \frac{1}{2}(x_{j+1} - x_{j-1}), \Delta y_j = \frac{1}{2}(y_{j+1} - y_{j-1}) \\ r &= \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}, \Gamma_j = \sqrt{(\Delta x_j)^2 + (\Delta y_j)^2} \end{aligned} \right\}$$

$i = j$ 時

由於 $kr \rightarrow 0$ 時， $H_1^{(1)}(kr) \rightarrow -i \frac{2}{\pi} \cdot \frac{1}{kr}$ ，故

$$H_{ij} = \int_{\Gamma_j} \frac{\partial}{\partial n} \left(-\frac{1}{4} H_0^{(1)}(kr) \right) d\Gamma$$



$$\begin{aligned} &\approx \frac{i}{4} \lim_{r \rightarrow 0} \left(-k H_1^{(1)}(kr) \cdot \frac{\partial r}{\partial n} \right) \Gamma_j \\ &= \frac{1}{2\pi} \Gamma_j \lim_{r \rightarrow 0} \frac{1}{r} \left(\frac{x_j - x_i}{r} \frac{\partial y}{\partial s} - \frac{y_j - y_i}{r} \frac{\partial x}{\partial s} \right) \\ &\approx \frac{1}{2\pi} \Gamma_j \lim_{r \rightarrow 0} \frac{1}{r} \left(\frac{x_j - x_i}{r} \frac{\Delta y}{\Delta s} - \frac{y_j - y_i}{r} \frac{\Delta x}{\Delta s} \right), (\Delta s = \Gamma_i) \end{aligned}$$

將 $x_j - x_i, y_j - y_i, \Delta x / \Delta s$ 及 $\Delta y / \Delta s$ 對 Δs 作Taylor展開，因 $i = j$ 時， $r \rightarrow 0$ ，即表示 $\Delta s \rightarrow 0$ ，得

$$\begin{aligned} H_{ii} &= \frac{1}{2\pi} \Gamma_i \cdot \frac{1}{2} (y_{ss} x_s - x_{ss} y_s)_i \\ &= \frac{1}{4\pi} (y_{ss} x_s - x_{ss} y_s)_i \Gamma_i \end{aligned}$$

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$kr \rightarrow 0$ 時， $H_0^{(1)}(kr) \rightarrow 1 + i \frac{2}{\pi} (\ln(kr/2) + \gamma)$ ，對元素取平均值，得

$$\begin{aligned} G_{ii} &= \frac{2}{\Gamma_i} \int_0^{\Gamma_i/2} \left[1 + i \frac{2}{\pi} \left(\ln \frac{kr}{2} + \gamma \right) \right] dr \cdot \Gamma_i \\ &= \frac{1}{\pi} \left[\gamma - 1 + \ln \frac{k\Gamma_i}{2} - i \frac{\pi}{2} \right] \cdot \Gamma_i \end{aligned}$$

$\gamma = 0.577216 \dots$ (Euler常數)



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