邊界積分方程式應用於Helmholtz方程式

函數 $f(x,y)$ 滿足下列Helmholtz方程式

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+k^2 f=0$$

依Helmholtz方程式基本解應用所述，對f(x,y)的基本解為 $-\frac{1}{4}H_0^{(1)}(Kr)$ 。將邊界線以m個一定元素加以離散，邊界面f與其法線方向導函數 $\overline{f}\,(=\partial f/\partial n)$ 的關係式如下

$$F = K^* \overline{F}$$

$$K^* = H^{-1}G$$

$$H = H_{ij} = \begin{cases} H_{ij} & i \neq j \\ H_{ij}+\frac{1}{2} & i = j \end{cases}$$

$$G = G_{ij}$$

i, j=1, 2, ⋯, m

H_{ij} 及 G_{ij} 數值計算如下

i≠j時

$$H_{ij} = \int_{\Gamma_j}\frac{\partial}{\partial n}\left(-\frac{1}{4}H_0^{(1)}(kr)\right)d\Gamma$$

$$\approx \frac{i}{4}kH_1^{(1)}(kr)\cdot\frac{\partial r}{\partial n}\Gamma_j$$

$$=\frac{i}{4}kH_1^{(1)}(kr)\left(\frac{\partial r}{\partial x}\frac{\partial x}{\partial n}+\frac{\partial r}{\partial y}\frac{\partial y}{\partial n}\right)\Gamma_j$$

$$=\frac{i}{4}kH_1^{(1)}(kr)\left(\frac{\partial r}{\partial x}\frac{\partial x}{\partial s}-\frac{\partial r}{\partial y}\frac{\partial y}{\partial s}\right)\cdot\Gamma_j$$

$$\approx\frac{i}{4}kH_1^{(1)}(kr)\left(\frac{x_j-x_i}{r}\Delta y_j-\frac{y_j-y_i}{r}\Delta x_j\right)$$

$$G_{ij}=\int_{\Gamma_j}\left(-\frac{1}{4}H_0^{(1)}(kr)\right)d\Gamma\approx-\frac{1}{4}H_0^{(1)}(kr)\cdot\Gamma_j$$

s表示切線

$$\left.\begin{aligned}&\Delta x_j=\frac{1}{2}\left(x_{j+1}-x_{j-1}\right),\Delta y_j=\frac{1}{2}\left(y_{j+1}-y_{j-1}\right)\\&r=\sqrt{(x_j-x_i)^2+(y_j-y_i)^2},\Gamma_j=\sqrt{(\Delta x_j)^2+(\Delta y_j)^2}\end{aligned}\right\}$$

i＝j時

由於k r→0時，$H_1^{(1)}(kr) \to -i\frac{2}{\pi}\cdot\frac{1}{kr}$，故

$$H_{ij} = \int_{\Gamma_j} \frac{\partial}{\partial n}\left(-\frac{1}{4}H_0^{(1)}(kr)\right)d\Gamma$$

$$\approx \frac{i}{4}\lim_{r\to 0}\left(-kH_1^{(1)}(kr)\cdot\frac{\partial r}{\partial n}\right)\Gamma_j$$

$$= \frac{1}{2\pi}\Gamma_j \lim_{r\to 0}\cdot\frac{1}{r}\left(\frac{x_j - x_i}{r}\frac{\partial y}{\partial s} - \frac{y_j - y_i}{r}\frac{\partial x}{\partial s}\right)$$

$$\approx \frac{1}{2\pi}\Gamma_j \lim_{r\to 0}\cdot\frac{1}{r}\left(\frac{x_j - x_i}{r}\frac{\Delta y}{\Delta s} - \frac{y_j - y_i}{r}\frac{\Delta x}{\Delta s}\right) \quad ,(\Delta s = \Gamma_i)$$

將 $x_j - x_i, y_j - y_i, \Delta x/\Delta s$ 及 $\Delta y/\Delta s$ 對 Δs 作Taylor展開，因i＝j時, r→0, 即表示 $\Delta s \to 0$，得

$$H_{ii} = \frac{1}{2\pi}\Gamma_i \cdot \frac{1}{2}(y_{ss}x_s - x_{ss}y_s)_i$$

$$= \frac{1}{4\pi}(y_{ss}x_s - x_{ss}y_s)_i \Gamma_i$$

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kr→0時，$H_0^{(1)}(kr) \to 1 + i\frac{2}{\pi}\left(\ln(kr/2) + \gamma\right)$，對元素取平均值，得

$$G_{ii} = \frac{2}{\Gamma_i}\int_0^{\Gamma_i/2}\left[1 + i\frac{2}{\pi}\left(\ln\frac{kr}{2} + \gamma\right)\right]dr \cdot \Gamma_i$$

$$= \frac{1}{\pi}\left[\gamma - 1 + \ln\frac{k\Gamma_i}{2} - i\frac{\pi}{2}\right]\cdot\Gamma_i$$

γ ＝0.577216……（Euler常數）

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